PROPAGATION OF ACOUSTIC WAVES IN A HEREDITARY ELASTIC MEDIUM

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The exponential-fractional functions introduced by Yu. N. Rabotnov [1] as kernels of integral operators have proved to be a highly effective tool when the Volterra principle is applied to the solution of static [1,2] and dynamic [3,4] problems in hereditary theory of elasticity. The reason for their effectiveness is that exponential-fractional kernels permit interpretation of the corresponding elastic operators on the basis of well defined rules. By investigating dissipative processes during harmonic deformations of such media, it is possible to establish the equivalence between the exponential-fractional kernels and the distribution functions of relaxation (retardation) constants [5,6]. As an example, the order of divisibility uniquely defines the radius of the complex-(compliance)-vector diagram [7] which constitutes clock diagrams of the type reported by Cole and Cole that correspond to a certain distribution function for relaxation (retardation) constants.

The analysis of dissipation processes may be based on the example of an acoustic wave which propagates in an unbounded medium and whose elastic operators are defined by exponential-fractional memory functions.

1. The system of equations of hereditary elasticity theory enables one to write the equations of motion in terms of the displacement vector u_i in the following form:

$$\rho \ddot{u_i} = \lambda u_{k,ki} + \mu(u_{k,ki}+u_{i,jj}), \tag{1.1}$$

where ρ is the density of the medium. The dot above the u denotes the time derivative, while the subscript behind the comma indicates spatial differentiation with respect to the corresponding coordinate. The elastic operators are defined as follows:

$$\lambda = \lambda\,(1+\Lambda^*), \qquad \Lambda^*\varepsilon = \int_0^\infty \Lambda\,(s)\,\varepsilon\,(t-s)\,ds\,,$$

$$\mu = \mu\,(1+M^*), \qquad M \;= \int_0^\infty M\,(s)\,\varepsilon\,(t-s)\,ds\,. \tag{1.2}$$

The solution of (1.1) will be sought in the form of an attenuating wave

$$u_i = A_i \quad \exp[i\omega t - (\alpha + i\omega\,/\,c)x_k\nu_k]\,, \tag{1.3}$$

where ν_i denotes a unit vector directed along the path of a wave characterized by a velocity $c > 0$, a frequency $\omega > 0$, an absorption coefficient $\alpha > 0$, and an amplitude A_i.

By substituting (1.3) into (1.1), we obtain

$$-\rho\omega^2 A_i = (\alpha + i\omega\,/\,c)^2 \times$$
$$\times\,\{A_k\nu_k\nu_i\,\lambda[1+\Lambda(\omega)] + (A_i + A_k\nu_k\nu_i)\mu[1+M(\omega)]\}\,, \tag{1.4}$$

$$M\,(\omega) = \int_0^\infty M\,(s)\,e^{-i\omega s}ds, \qquad \Lambda\,(\omega) = \int_0^\infty \Lambda\,(s)\,e^{-i\omega s}\;s\,. \tag{1.5}$$

From relation (1.4) it is possible to determine the wave velocity c, the absorption coefficient α, and the logarithmic decrement δ which defines the attenuation of a wave in space. One must distinguish here between two types of wave: transverse and longitudinal waves propagating independently at the velocities c_t and c_l, respectively.

The characteristics of a transverse wave are determined from Eq. (1.4) by setting $A_k\nu_k = 0$ in it; we obtain

$$-\rho\omega^2 = (\alpha_t + i\omega\,/\,c_t)\mu[1+M(\omega)]. \tag{1.6}$$

Hence, we get

$$\rho c_t^2 = \mu^\circ \sec^2 {}^1\!/_2\varphi_t, \quad \mu^\circ = \mu\,|\,1 + M(\omega)|\,, \tag{1.7}$$

$$\alpha_t = \omega c_t^{-1}\ \mathrm{tg}\ {}^1\!/_2\,\varphi_t, \ \ \delta = 2\pi\ \ \mathrm{tg}\,{}^1\!/_2\varphi_t\,, \tag{1.8}$$

$$\mathrm{tg}\ \varphi_t = \frac{\mathrm{Im}\,[1+M\,(\omega)]}{\mathrm{Re}\,[1+M\,(\omega)]}\,, \qquad 0 \leqslant \varphi_t \leqslant \frac{1}{2}\pi\,. \tag{1.9}$$

where μ° is the absolute value of the complex modulus, which may be assumed to define the dynamic modulus; and tg φ_t is the conventional mechanical-loss tangent for the one-dimensional case.

For a longitudinal wave, $A_k\nu_k \neq 0$. Then, multiplying relation (1.4) by ν_i, and summing over the recurrent subscripts, we obtain

$$-\rho\omega^2 = (\alpha_l + i\omega c_l^{-1})^2\,\{\lambda\,[1+\Lambda\,(\omega)] + 2\mu\ \ [1\ \text{sic}] \tag{1.10}$$

From here, by analogy to the transverse wave, we have

$$\rho e^2 = N\ \ \sec^2 {}^1\!/_2\varphi_l, \tag{1.11}$$

$$N = |\,N^*\,| = |\,\lambda[1+\Lambda(\omega)] + 2\mu[1+M(\omega)]\,|\,,$$
$$\alpha_l = \omega c_l^{-1}\ \ \mathrm{tg}\,{}^1\!/_2\varphi_l, \qquad \delta = 2\pi\mathrm{tg}\,{}^1\!/_2\varphi_l\,, \tag{1.12}$$

$$\mathrm{tg}\varphi_l = \mathrm{Im}N^*\,/\,\mathrm{Re}N^*, \qquad 0 \leqslant \varphi_l \leqslant {}^1\!/_2\pi\,. \tag{1.13}$$

For convenience, the longitudinal wave characteristics will by expressed through elastic hydrostatic stress and shear operators. This can be achieved with the aid of the relation

$$N^* = K[1 + K^*(\omega)] + {}^4\!/_3\mu[1 + M^*(\omega)]. \tag{1.14}$$

On the basis of the Fourier transform theorems

$$\lim_{\omega\to\infty}\ \Lambda(\omega) = \lim_{\omega\to\infty} \mathrm{M}(\omega) = 0, \lim_{\omega\to\infty} \omega\Lambda(\omega) = -\,i\Lambda(0),$$
$$\lim_{\omega\to\infty} \omega M(\omega) = -\,iM(0)\,, \tag{1.15}$$

from relations (1.7), (1.8), (1.11) for $\omega \to \infty$, we obtain

$$\rho c_{t\infty}{}^2 = \mu_\infty, \qquad \rho c_{l\infty}{}^2 = K_\infty + {}^4\!/_3\,\mu_\infty\,, \tag{1.16}$$
$$\alpha_{t\infty}c_{t\infty} = {}^1\!/_2\,M\,(0), \qquad \alpha_{l\infty}c_{l\infty} = {}^1\!/_2\,[\lambda\Lambda\,(0) + 2\mu M\,(0)]\ . \tag{1.17}$$

Hence, for $\omega \to \infty$, the acoustic wave velocities are equal to the corresponding "nonrelaxed" elastic velocities, while the absorption coefficients are defined by hereditary kernels taken for $t = 0$, i.e., for all weakly singular kernels at $\omega \to \infty$, α_t and $\alpha_l \to \infty$. For $\omega \to 0$, with allowance for the normalization condition

$$\int_0^\infty \Lambda\,(s)\,ds = \int_0^\infty M\,(s)\,ds = 1 \tag{1.18}$$

we get

$$\rho c_{t_0}{}^2 = \mu_0, \ \ \rho c_{l_0}{}^2 = K_0 + {}^4\!/_3\mu_0, \ \ \alpha_{t_0} = \alpha_{l_0} = 0 \tag{1.19}$$

This means that the acoustic wave velocities become equal to the corresponding "relaxed" elastic velocities, while absorption is discontinued.

2. Let us first examine a transverse wave for the case in which the kernel of the operator M(t) is an exponential-fractional function

$$M\,(t) = -\,\varkappa \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma\,[(\gamma\,(n+1)]} \left(\frac{t}{\tau_\mu}\right)^{\gamma(n+1)-1},$$

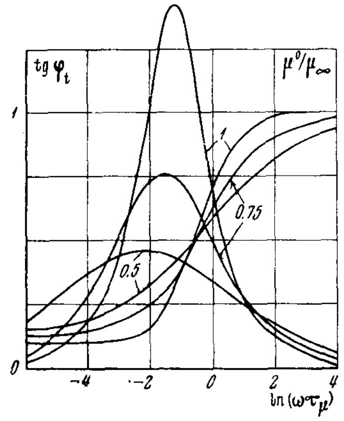

Fig. 1

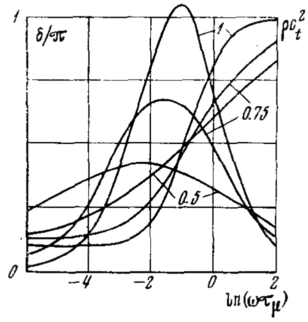

Fig. 2

$$\rho c_t{}^2 = qh^{-1}\sec\varphi_t\sec^2{}^1/_2\varphi_t\,, \tag{2.9}$$

$$\alpha_t = \omega(\rho\,/\,\mu^0)^{1/2}\sin{}^1/_2\varphi_t,\ \ \delta = 2\pi\,\mathrm{tg}{}^1/_2\varphi_t\,. \tag{2.10}$$

$$\varkappa = \Delta\mu\,/\,\mu_\infty\tau_\mu,\ \ \Delta\mu = \mu_\infty - \mu_0 \tag{2.1}$$

where μ_∞ and μ_0 are the nonrelaxed and relaxed values of the shear modulus, respectively; τ_μ is the relaxation time of shear stresses; and γ is the divisibility parameter. For $\gamma = 1$, the kernel (2.1) degenerates to an ordinary exponential function, while the hereditary shear characteristics are described by a standard linear model. Substituting (2.1) into (1.5), we obtain for the mechanical-loss tangent (1.9) and the dynamic modulus (1.7), respectively:

$$tg\ \varphi_t = \frac{\Delta\mu\sin\psi}{g},\ \ \psi = \frac{\pi\gamma}{2}, \tag{2.2}$$

$$g \equiv \frac{\mu_0}{(\omega\tau_\mu)^\gamma} + \mu_\infty(\omega\tau_\mu)^\gamma + (\mu_0 + \mu_\infty)\cos\psi\ \ \mu^0 = \frac{g\sec\varphi_t}{h},$$

$$h \equiv \frac{1}{(\omega\tau_\mu)^\gamma} + (\omega\tau_\mu)^\gamma + 2\cos\psi\,. \tag{2.3}$$

It can be readily shown that for condition $(\omega\tau_\mu)^{2\gamma} = \mu_0/\mu_\infty$, the mechanical-loss tangent has its maximum value

$$\mathrm{tg}\,\varphi_m = \frac{\Delta^\pm\sin\psi}{2 + \Delta^\pm\cos\psi},\ \ \ \Delta^\pm = \frac{\mu_\infty \pm \mu_0}{\sqrt{\mu_\infty\mu_0}}\,. \tag{2.4}$$

This can be clearly seen from Fig. 1, where γ has been chosen as the parameter and its values are indicated by numbers on the curves $\mu_0/\mu_\infty = 0.1$. The figure shows that for a standard linear body ($\gamma = 1$) the dynamic modulus $\mu^\circ(\omega)$ attains its limiting values $\mu^\circ(\omega \to 0) = \mu_0$ and $\mu^\circ(\omega \to \infty) = \mu_\infty$ much faster than $\mathrm{tg}\,\varphi_t \to 0$ does for $\omega \to 0$ and $\omega \to \infty$. This can be explained by the difference in the asymptotic behavior of these quantities which for $\gamma = 1$ are illustrated by the following relations [8]:

$$\mu^0 = \mu_\infty - \frac{\mu_\infty{}^2 - \mu_0{}^2}{2\mu_\infty\omega^2\tau_\mu{}^2},\ \ \ \mathrm{tg}\,\varphi_t\ \ \ \frac{\Delta\mu}{\mu_\infty\,\omega\tau_\mu},\ \ \ \omega\tau_\mu \gg 1 \tag{2.5}$$

$$\mu^0 = \mu_0 + \frac{\mu_\infty{}^2 - \mu_0{}^2}{2\mu_0}\,\omega^2\tau_\mu{}^2,\ \ \ \mathrm{tg}\,\varphi_t = \omega\tau_\mu\,\frac{\Delta\mu}{\mu_0},\ \ \ \omega\tau_\mu \ll 1 \tag{2.6}$$

This difference vanishes gradually as γ decreases, since the nature of the asymptotic behavior of the loss tangent and dynamic modulus changes

$$\mu^0 = \mu_\infty - \frac{\Delta\mu\cos\psi}{\omega^\gamma\,\tau_\mu{}^\gamma},\ \ \ \mathrm{tg}\,\varphi_t = \frac{\Delta\mu\sin\psi}{\mu_\infty\omega^\gamma\tau_\mu{}^\gamma},\ \ \ \omega\tau_\mu \gg 1, \tag{2.7}$$

$$\mu^0 = \mu_0 + \Delta\mu(\omega\tau_\mu)^\gamma\cos\psi,$$

$$\mathrm{tg}\varphi_t = \Delta\mu\mu_0{}^{-1}(\omega\tau_\mu)^\gamma\sin\psi,\ \ \omega\tau_\mu \ll 1\,. \tag{2.8}$$

If $\mathrm{tg}\,\varphi_t$ and μ° are known, from formulas (1.7) and (1.8) it is not difficult to determine the velocity, the absorption coefficient and the logarithmic decrement of the transverse wave,

Figure 2 shows the curves $\rho c_t^2 = f(\ln\omega\tau_\mu)$ and $\delta = f(\ln\omega\tau_\mu)$. It is obvious that the behavior of the squared velocity differs from that of the logarithmic decrement and the behavior of the dynamic modulus differs from that of the loss tangent. This difference vanishes, however, with decreasing attenuation, whose magnitude is defined by three factors: the frequency range $\omega\tau_\mu$, the degree of relaxation μ_0/μ_∞, and the parameter γ. The values of γ are indicated by numbers at the curves. For fixed μ_0/μ_∞ and γ, the asymptotic behavior of ρc_t^2 and δ/π for $\omega\tau_\mu \to \infty$ and $\omega\tau_\mu \to 0$ is defined, respectively, by expressions (2.7) and (2.8). For fixed $\omega\tau_\mu$ and μ_0/μ_∞, the difference between $\mathrm{tg}\,\varphi_t$ and δ/π and between μ° and ρc_t^2, vanishes in the case of $\gamma \to 1$ (standard linear body). For given $\omega\tau_\mu$ and γ, the following limiting relations are valid:

$$\lim_{\xi\to1}\mathrm{tg}\,\varphi_t = \lim_{\xi\to1}\delta = 0,\ \ \ \xi \equiv \mu_0\,/\,\mu_\infty,$$

$$\lim_{\xi\to0}\mathrm{tg}\,\varphi_t = [\cos\psi + (\omega\tau_\mu)^\gamma]^{-1}\sin\psi\,,$$

$$\lim_{\xi\to1}(\delta\,/\,\pi) = \{\cos\psi + (\omega\tau_\mu)^\gamma +$$

$$+ [1 + 2(\omega\tau_\mu)^\gamma\cos\psi + (\omega\tau_\mu)^{2\gamma}]^{1/2}\}^{-1}\sin\psi\,,$$

$$\lim_{\xi\to1}\mu^0 = \mu_\infty = \rho c_\infty{}^2{}_t\,,$$

$$\lim_{\xi\to0}\mu^0 = \mu_\infty\,[1 + 2(\omega\tau_\mu)^{-\gamma}\cos\psi + (\omega\tau_\mu)^{-2\gamma}]^{-1/2}\,,$$

$$\lim_{\xi\to0}\rho c_t{}^2 = 2\mu_\infty\,[1 + (\omega\tau_\mu)^{-\gamma}\cos\psi + (\mu_\infty{}^{-1}\lim_{\xi\to0}\mu^0)^{-1}]^{-1}. \tag{211}$$

When all the values vary simultaneously, attenuation has its peak value for the condition $(\omega\tau_\mu)^2 = \mu_0\,/\,\mu_\infty$. Hence, the difference between $\mathrm{tg}\,\varphi_t$ and δ/π and between μ^0 and ρc_t^2 is maximum. Figure 3 gives a comparison between $\mathrm{tg}\,\varphi_{mt}$ (curves a) and $(\delta\,/\,\pi)_m$ (curves b). The numbers indicate the value of parameter γ. When the ratio $\mu_0\,/\,\mu_\infty$ decreases, the difference between these two quantities increases from 0 to $\mathrm{tg}\,\psi - 2\,\mathrm{tg}\,(\psi/2)$ for $0 < \gamma < 1$, and tends to infinity in the case of a standard linear body ($\gamma = 1$).

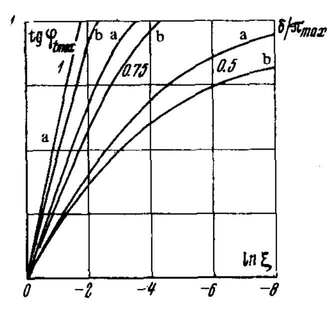

Fig. 3

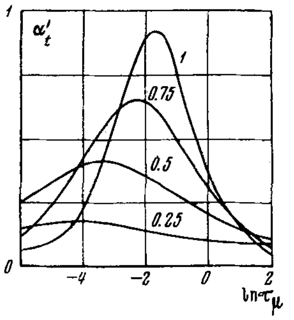

Fig. 4

In formulas (2.2)–(2.4), the frequency dependence (variance) of the respective quantities is defined by a dimensionless parameter $\omega\tau_\mu$, which depends not only on the frequency but on the temperature as well, since, according to the Arrhenius law, $\tau_\mu = \tau_0 \exp(u/kt)$. An increase in temperature, therefore, is equivalent to a decrease in frequency; this fact is used in experimental dispersion studies. This equivalence does not apply to the absorption coefficient $\alpha'_t = \alpha_t c_{t\infty}\omega^{-1}$, since α'_t attains a maximum as a function of the temperature (Fig. 4), while ω varies monotonically (Fig. 5). The influence of γ (whose values are indicated by numbers at the curves) is different in each case which is a result of the violation of the temperature-frequency equivalence.

This can be seen particularly clearly from an analysis of the asymptotic behavior of the coefficient α_t,

$$\alpha_t = \frac{\Delta\mu\,\omega\sin\psi}{2c_{l\infty}\omega^{\gamma}\tau_\mu^{\ \gamma}}, \qquad \omega\tau_\mu \gg 1, \tag{2.12}$$

$$\alpha_t = \frac{\Delta\mu\omega^{\gamma+1}\tau_\mu^{\ \gamma}\sin\psi}{2c_{t_0}\mu_0}, \qquad \omega\tau_\mu \ll 1. \tag{2.13}$$

For ω = const, when $\tau_\mu \to 0$ and $\tau_\mu \to \infty$, the absorption coefficient α_t' tends to zero (Fig. 4). For τ_μ = const, if ω tends to zero α_t' tends to zero for any $\gamma \in (0.1]$. If, however, $\omega \to \infty$, then $\alpha_t' \to \infty$ for $\gamma \neq 1$, while $\alpha_{l\infty}' = \Delta\mu / 2c_{t\infty}\tau_\mu$ for $\gamma = 1$ (Fig. 4).

3. Let us determine the characteristics of a longitudinal wave, assuming that the volume relaxation kernel is also described by a Rabotnov exponential-fractional function. Then,

$$\operatorname{tg}\varphi_l = \frac{\Delta K h_K^{-1}\sin\psi_K + {}^4/_3\,\Delta\mu h_\mu^{-1}\sin\psi_\mu}{g_K h_K^{-1} + {}^4/_3\, g_\mu h_\mu^{-1}}, \tag{3.1}$$

$$N = (g_K h^{-1} + {}^4/_3\, g_\mu h_\mu^{-1})\sec\varphi_l. \tag{3.2}$$

The subscript K means that the corresponding quantities characterize volume relaxation.

The influence of volume relaxation on shear relaxation can be assessed from formulas (3.1)–(3.3). The theoretical possibility of such an influence was discussed in [9] for the case $\gamma = \gamma_\mu = \gamma_K$. For given relaxation characteristics τ_μ, τ_K, μ_0/μ_∞, K_0/K_∞ and a given nonrelaxed Poisson ratio $\nu_\infty = 0.3$, the volume relaxation peak appears most clearly in the case $\gamma = 1$, i.e., when the hereditary properties of the shear and bulk moduli are described by a standard linear model.

Since with decreasing γ there occurs a "spreading" of the spectrum, volume relaxation manifest itself less strongly, until it vanishes altogether. This is indicated by the nature of the asymptotic behavior of tg φ_l and N.

In the case of a standard linear body $\gamma_\mu = \gamma_K = \gamma = 1$, we have

$\omega \gg 1$

$$\operatorname{tg}\varphi_l = \frac{1}{K_\infty + {}^4/_3\,\mu_\infty}\left(\frac{\Delta K}{\omega\tau_K} + \frac{4}{3}\,\frac{\Delta\mu}{\omega\tau_\mu}\right),$$

$$N = K_\infty + \frac{4\mu_\infty}{3} - \frac{1}{K_\infty + {}^4/_3\,\mu_\infty}\left\{\frac{8\Delta\mu\,[K_\infty + {}^1/_3\,(\mu_\infty + \mu_0)]}{3\omega^2\tau_\mu^{\ 2}} + \right.$$

$$\left. + \frac{\Delta K}{2(\omega\tau_K)^2}\left[K_\infty + K_0 + \frac{16\mu_\infty}{3}\right] - \frac{\Delta K\Delta\mu}{3\omega^2\tau_\mu\tau_K}\right\}, \tag{3.3}$$

$\omega \ll 1$

$$\operatorname{tg}\varphi_l = (K_0 + {}^4/_3\,\mu_0)^{-1}(\Delta K\omega\tau_K + {}^4/_3\,\Delta\mu\omega\tau_\mu),$$

$$N = K_0 + {}^4/_3\,\mu_0 +$$

$$+ (K_0 + {}^4/_3\,\mu_0)^{-1}\{{}^8/_3\,\Delta\mu\,(\omega\tau_\mu)^2\,[K_0 + {}^1/_3\,(\mu_0 + \mu_\infty)] +$$

$$+ {}^1/_2\Delta K\,(\omega\tau_K)^2\,(K_0 + K_\infty + {}^{16}/_3\,\mu_0) + 4\,\Delta K\Delta\mu / 3\omega^2\tau_\mu\tau_K\}. \tag{3.4}$$

When γ decreases, the nature of the asymptotic behavior of the loss tangent and dynamic modulus changes,

$\omega \gg 1$

$$\operatorname{tg}\varphi_l = \frac{1}{K_\infty + {}^4/_3\,\mu_\infty}\left[\frac{\Delta K}{(\omega K_K)^\gamma} + \frac{4\Delta\mu}{3(\omega\tau_\mu)^\gamma}\right]\sin\psi,$$

$$N = K_\infty + \frac{4\mu_\infty}{3} - \left[\frac{\Delta K}{(\omega\tau_K)^\gamma} + \frac{4\Delta\mu}{3(\omega\tau_\mu)^\gamma}\right]\cos\psi, \tag{3.5}$$

$\omega \ll 1$

$$\operatorname{tg}\varphi_l = (K_0 + {}^4/_3\,\mu_0)^{-1}\,[\Delta K\,(\omega\tau_K)^\gamma + {}^4/_3\,\Delta\mu\,(\omega\tau_\mu)^\gamma]\sin\psi,$$

$$N = K_0 + {}^4/_3\,\mu_0 + [\Delta K\,(\omega\tau_K)^\gamma + {}^4/_3\,\Delta\mu\,(\omega\tau_\mu)^\gamma]\cos\psi. \tag{3.6}$$

In the analysis of the aymptotic behavior of the coefficient α_l, one can observe an impairment of the temperature-frequency equivalence

$\omega \gg 1$

$$\alpha_l = {}^1/_2\,\omega\rho^{1/2}\,(K_\infty + {}^4/_3\,\mu_\infty)^{-3/2} \times$$

$$\times [\Delta K\,(\omega\tau_K)^{-\gamma} + {}^4/_3\,\Delta\mu\,(\omega\tau_\mu)^{-\gamma}]\sin\psi, \tag{3.7}$$

$\omega \ll 1$

$$\alpha_l = {}^1/_2\,\omega\rho^{1/2}\,(K_0 + {}^4/_3\,\mu_0)^{-3/2} \times$$

$$\times [\Delta K\,(\omega\tau_K)^\gamma + {}^4/_3\,\Delta\mu\,(\omega\tau_\mu)^\gamma]\sin\psi. \tag{3.8}$$

For ω = const, when $\tau_\mu \to 0$ and $\tau_\mu \to \infty$, the absorption coefficient α_l tends to zero. For τ_μ = const, when $\omega \to 0$, α_l tends to zero for all $\gamma \in (0.1]$. However, when $\omega \to \infty$, then $\alpha_l \to \infty$ for $\gamma \neq 1$ and $\alpha_l \to$ const for $\gamma = 1$.

Thus, by using exponential-fractional functions as kernels of elastic integral operators, it is possible to study all the characteristics of acoustic wave propagation in a viscoelastic medium with a symmetrical relaxation spectrum, since to an exponential-fractional kernel there corresponds a symmetrical distribution function of the relaxation frequencies.

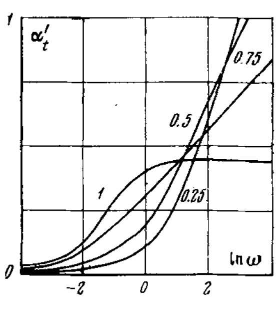

Fig. 5

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